International Journal of Theoretical Physics, Vol. 38, No. 7, 1999

$U_q[sl(2)]$ Quantum Algebra in Quantum Hall Effect

A. Jellal[1]

Received December 24, 1998

For a two-dimensional system of electrons described by a Hamiltonian involving two- and three-body interactions and an external transverse magnetic field, we construct the $U_q[sl(2)]$ quantum algebra, where the deformation parameter q is related to the filling factor ν. We show that the Laughlin states form a representation of this algebra.

The $U_q[sl(2)]$ quantum algebra has its origin in the inverse scattering method (Fadeev, 1984; Kulish and Sklyanin, 1982) and the first such structure, i.e., $U_q[sl(2)]$, appeared in studies of the Yang–Baxter equation (Kulish and Reshetkhin, 1981, 1982, 1983a, b). Subsequent developments have shown that the Hopf algebra description of quantum algebras is the appropriate one (Drinfeld, 1986; Mansour, 1998; Reshetkhin *et al.*, 1989). Also an extension of the theory of quantum algebras to supersymmetric quantum Lie algebra has been achieved (Chaichain and Kulish, 1990; Kulish, 1989; Kulish and Reshetkhin, 1989). The representation of this quantum algebra was applied to formulate the Bethe anzatz for the problem of Bloch electrons in a magnetic field, i.e., the Azbel–Hofstadter problem (Fadeev and Kashaev, 1993; Weigmann and Zabrodin, 1993a, b). Naturally, these symmetry are realized also in the Maxwell–Chern–Simons (MCS) theory, in the pure Chern–Simons theory (CS) on the torus, in the Landau problem, and in the quantum Hall effect (Alimohammadi and Shafei Deh Abad, 1996; Kogan, 1994; Sato, 1994), where the latter emerges in a two-dimensional system of electrons in the presence of a strong perpendicular uniform magnetic field B (Prange and Girvin, 1990; Stone, 1992). It is characterized by the existence of a series

[1] Laboratoire de Physique Théorique (LPT-ICAC), Faculté des Sciences, Rabat, Morocco; e-mail: oudrhiri@fsr.ac.ma.

0020-7748/99/0700-189 3$16.00/0

of plateaus where the Hall conductivity is quantized and the longitudinal conductivity vanishes.

The main objective of this paper is to show how the introduction of the specified interactions of two-body and three-body types between particles in the Hamiltonian involving the electrons in an external magnetic field leads to the realization of $U_q[sl(2)]$. We find the Laughlin states are a representation of it.

To start let us consider the following N-body Hamiltonian of electrons confined in a two-dimensional plane (x, y) in the presence of a uniform magnetic field B perpendicular to the plane and with specified interactions between particles in the complex notation ($\hbar = c = m = e = 1$, $B = 2$) (Ghosh and Rao, 1997)

$$
\begin{aligned}
H = & \sum_{i=1}^{N}(-4\partial_i\bar{\partial}_i + z_i\partial_i - \bar{z}_i\bar{\partial}_i + z_i\bar{z}_i) \\
& + 4\eta \sum_{i\neq j}^{N} \left(\frac{1}{z_{ij}}\left(\bar{\partial}_i - \frac{z_i}{2}\right) - \frac{1}{\bar{z}_{ij}}\left(\partial_i + \frac{\bar{z}_i}{2}\right)\right) \\
& + 4\eta^2 \sum_{i,j\neq i,i\neq k}^{N} \left(\frac{1}{z_{ij}\bar{z}_{ik}}\right)
\end{aligned} \tag{1}
$$

where we have taken the vector potential A in a symmetric gauge ($A_z = -iz$, $A_{\bar{z}} = i\bar{z}$), $z_i = x_i + iy_i$ denotes the ith position of the particles, $z_{ij} = z_i - z_j$, $\partial_i = \partial/\partial_{z_i}$, and η is an odd integer. In (1), the first term represents the quasi-canonical momentum $\pi = (P - A)$, the second is the two-body interactions, and the third is the three-body interactions between particles.

Let us now define the annihilation a_i and creation a_i^+ operators by the following expressions:

$$
a_i = \frac{1}{2}\left(-2\partial_i + \bar{z}_i - 2\eta \sum_{j\neq i}^{N} \frac{1}{z_{ij}}\right) \tag{2}
$$

$$
a_i^+ = \frac{1}{2}\left(-2\bar{\partial}_i - z_i - 2\eta \sum_{j\neq i}^{N} \frac{1}{\bar{z}_{ij}}\right) \tag{3}
$$

where $[\partial_i, z_j] = [\bar{\partial}_i, z_j] = \delta_{ij}$. They satisfy the commutation relations

$$
[a_i, a_j^+] = \delta_{ij} \tag{4}
$$

$$
[a_i, a_j] = [a_i^+, a_j^+] = 0 \tag{5}
$$

The Hamiltonian (1) is given now as a function of the operators a_i and a_i^+ by

$$H = \sum_{i=1}^{N} (a_i^+ \, a_i + a_i \, a_i^+) \tag{6}$$

It is convenient to introduce the operators b_i and b_i^+, which we will use below, as follows:

$$b_i = \frac{1}{2}\left(-2\bar{\partial}_i + z_i - 2\eta \sum_{j \neq i}^{N} \frac{1}{\bar{z}_{ij}} \right) \tag{7}$$

$$b_i^+ = \frac{1}{2}\left(-2\bar{\partial}_i - \bar{z}_i - 2\eta \sum_{j \neq i}^{N} \frac{1}{z_{ij}} \right) \tag{8}$$

They obey the commutation relations

$$[b_i, b_j^+] = \delta_{ij} \tag{9}$$

$$[b_i, b_j] = [b_i^+, b_j^+] = [a_i, b_j] = [a_i, b_j^+] = 0 \tag{10}$$

Now let us investigate the possibility of realizing sine or w_∞-symmetry from the operators b_i and b_i^+. To begin, let us present the following operators for a given pair (n_1, n_2) (Kogan, 1994):

$$T^i_{(n_1,n_2)} = e^{n_1 b_i + n_2 b_i^+}, \qquad n_1, n_2 \in C \tag{11}$$

It is not difficult to see that the operators $T^i_{(n_1,n_2)}$ and $T^j_{(m_1,m_2)}$ satisfy the relations

$$T^i_{(n_1,n_2)} \, T^j_{(m_1,m_2)} = e^{n_1 b_i + n_2 b_i^+ + m_1 b_j + m_2 b_j^+} \; e^{\delta_{ij}(n_1 m_2 - n_2 m_1)} \tag{12}$$

One sees that for $i = j$, the above relation becomes

$$T^i_{(n_1,n_2)} \, T^i_{(m_1,m_2)} = T^i_{(n_1+m_1,n_2+m_2)} \; e^{(n_1 m_2 - n_2 m_1)} \tag{13}$$

From (13), it is easy to check the operators $T^i_{(n_1,n_2)}$ satisfying the commutations

$$[T^i_{(n_1,n_2)}, T^i_{(m_1,m_2)}] = 2i \sin \frac{i}{2} (n_1 m_2 - n_2 m_1) \, T^i_{(n_1+m_1,n_2+m_2)} \tag{14}$$

Here we require the following condition over $(n_1 m_2 - n_2 m_1)$ to be pure imaginary. This is exactly the sine algebra or w_∞-symmetry (Fairlie *et al.*, 1989, 1990; Fairlie and Zachos, 1989), which is the deformation à la Moyal of the Lie algebra $C^\infty(T^2)$ of a function on the two-dimensional torus.

Now we can realize the quantum algebra $U_q[sl(2)]$. First, let us recall that this quantum algebra is defined by four generators E^+, E^-, k, and k^{-1}

which obey the following commutation relations (Drinfeld, 1986; Sklyanin, 1991):

$$[E^{+}, E^{-}] = \frac{k^2 - k^{-2}}{q - q^{-1}} \tag{15}$$

$$kE^{\pm} k^{-1} = q^{\pm 1} E^{\pm} \tag{16}$$

where q is the so-called deformation parameter. These generators can be constructed by combining the operators given by equation (11). Precisely, let us consider the following construction depending on two arbitrary noncollinear pairs (n_1, n_2), $(-n_1, n_2)$ and for a fixed i:

$$E_i^{+} = \frac{T^i_{(n_1,n_2)} - T^i_{(-n_1,n_2)}}{q - q^{-1}} \tag{17}$$

$$E_i^{-} = \frac{T^i_{(-n_1,-n_2)} - T^i_{(n_1,-n_2)}}{q - q^{-1}} \tag{18}$$

$$k_i = T^i_{(n_1,0)}, \qquad k_i^{-1} = T^i_{(-n_1,0)} \tag{19}$$

Calculating the commutation relations between these generators, we recover $U_q[sl(2)]$ if the deformation parameter is chosen to be

$$q = e^{n_1 n_2} \tag{20}$$

At this step we note that one can construct the sine algebra for N electrons. For this, we define the total symmetry operator by the product of N copies of one-particle operators such as

$$T_{(n_1,n_2)} = \exp\left[\sum_{i=1}^{N} (n_1 b_i + n_2 b_i^{+})\right] \tag{21}$$

which satisfy the following commutation relations:

$$[T_{(n_1,n_2)}, T_{(m_1,m_2)}] = 2i \sin \frac{iN}{2} (n_1 m_2 - n_2 m_1)\, T_{(n_1+m_1,n_2+m_2)} \tag{22}$$

In the same way, we can construct $U_q[sl(2)]$ as in (17)–(19)

$$E^{+} = \frac{T_{(n_1,n_2)} - T_{(-n_1,n_2)}}{q - q^{-1}} \tag{23}$$

$$E^{-} = \frac{T_{(-n_1,-n_2)} - T_{(n_1,-n_2)}}{q - q^{-1}} \tag{24}$$

$$k = T_{(n_1,0)}, \qquad k^{-1} = T_{(-n_1,0)} \tag{25}$$

In this situation, the deformation parameter q is defined as

$$q = e^{Nn_1n_2} \tag{26}$$

Now let us discuss this result. With the appropriate choice of the complex numbers n_1 and n_2, such as

$$n_1 = \frac{2\pi}{L_x}, \qquad n_2 = \frac{i\pi}{L_y} \tag{27}$$

the q-deformation parameter given by equation (26) can be written as

$$q = e^{2i\pi\nu}, \qquad \nu = \frac{N\pi}{L_xL_y} \tag{28}$$

where ν is defined as the number of electrons N per of degeneracy number of the Landau level $eBL_xL_u/2\pi\hbar c$ (in our case $L_xL_y/2\pi$) (Prange and Girvin, 1990), with L_x and L_y defining the size of the two-dimensional system of electrons along the x axis and y axis, respectively. Hence, this equation leads to a possible relation between the q-deformation parameter and the filling factor ν, especially in the case where q is a root of unity, namely

$$q = e^{2i\pi/l}, \quad l \in N^* \tag{29}$$

When l takes only odd values, $l \equiv \eta$, then by comparing equations (28) and (29) we can derive the series for the filling factor $\nu = 1/\eta$ (η odd integer)(Fröl-ich and Zee, 1991; Jellal, 1998).

Now let us turn to the quantum algebra structure on some basis of many-particle wave functions. For convenience, we will focus on the Laughlin wave functions (Laughlin, 1983; Prange and Girvin, 1990).

For a given filling factor $\nu = 1/\eta$ (η odd integer), the ground-state wave function is described very accurately by the variational wave functions proposed by Laughlin (1983)

$$\psi_\eta(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = \prod_{i<j} (\bar{z}_i - \bar{z}_j)^\eta \exp\left(-\frac{1}{2}\sum_{i=1}^{N} z_i, \bar{z}_i \right) \tag{30}$$

Now we denote the wave functions $\psi_\eta(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N)$ by $\psi_\eta(z_i)$, $i = 1, 2, \ldots, N$. With the use of equation (21), we can verify that these operators and the wave functions given above satisfy

$$T_{(n_1,n_2)}\, \psi_\eta(z_i) = e^{-Nn_1n_2/2}\, \psi_\eta(z_i - 2n_2) \tag{31}$$

where $\psi_\eta(z_i - 2n_2) = \psi_\eta(z_1 - 2n_2, \bar{z}_n, \ldots, z_N - 2n_2, \bar{z}_n)$. Then we obtain the action of the quantum algebra on the wave functions (30) using equations (31) and (23)–(25),

$$E^{\pm}\ \psi_\eta(z_i) = \left[-\frac{1}{2}\right]_q \psi_\eta(z_i - 2n_2) \tag{32}$$

$$k^{\pm 1}\ \psi_\eta(z_i) = \psi_\eta(z_i) \tag{33}$$

These relations show that the Laughlin wave functions form a representation of the quantum algebra $U_q[sl(2)]$.

In this paper, we have realized the quantum algebra $U_q[sl(2)]$ of an explicit model for electrons in an external magnetic field and with specified interactions between electrons. We have also shown that the Laughlin wave functions form a representation basis of this quantum algebra whose deformation parameter is related to the filling factor. In the special case where q is a root of unity, we have recovered the series $\nu = 1/\eta$ (η odd integer) characterizing the fractional quantum Hall effect.

ACKNOWLEDGMENTS

I gratefully acknowledge helpful discussions with Prof. A. El Hassouni and also useful comments on the manuscript. I also thank Dr. E. H. El Kinani for reading the manuscript.

REFERENCES

Alimohammadi, M., and Shafi Deh Abad, A. (1996). *Journal of Physics: Mathematical and General*, **29A**, 551.
Chaichain, M., and Kulish, P. P. (1990). *Physics Letters*, **234B**, 72.
Drinfeld, V. G. (1986). Quantum groups, in *Proceeding of International Congress of Mathematicians*, Berkeley, California.
Fadeev, L. D. (1984). *Les houches Lectures*, Elsevier, Amsterdam.
Fadeev, L. D., and Kashaev, R. M. (1993). hep-th/9312133, Preprint.
Fairlie, D. B., and Zachos, C. K. (1989). *Physics Letters*, **244B**, 101.
Fairlie, D. B., Flecher, P., and Zachos, C. K. (1989). *Physics Letters*, **218B**, 203.
Fairlie, D. B., Flecher, P., and Zachos, C. K. (1990). *Journal of Mathematical Physics*, **31**, 1088.
Frölhich, J., and Zee, A. (1991). *Nuclear Physics*, **364B**, 564.
Ghosh, R. K., and Rao, S. (1998). *International Journal of Modern Physics,* **12B**, 1125.
Jellal, A. (1998). *International Journal of Theoretical Physics*, **37**, 2751.
Jimbo, M. (1985). *Letters in Mathematical Physics*, **10**, 63.
Jimbo, M. (1986). *Letters in Mathematical Physics*, **11**, 247.
Kogan, I. I. (1994). *International Journal of Modern Physics*, **9A**, 3889.
Kulish, P. P. (1989). Preprint RIMS-615 Kyoto; in *Proceeding of Alushta School on QHTE*, Springer, Berlin.
Kulish, P. P., and Reshetkhin, N. Y. (1981). *Zap. Nauch. Semin. LOMI*, **101** [in Russian].

Kulish, P. P., and Reshetkhin, N. Y. (1982). *Functional Analysis and Its Applications*, **16**, 27.
Kulish, P. P., and Reshetkhin, N. Y. (1983a). *Soviet Journal of Mathematics*, **23**, 2345.
Kulish, P. P., and Reshetkhin, N.Y. (1983b). *Functional Analysis and Its Applications*, **17**, 34.
Kulish, P. P., and Reshetkhin, N. Y. (1989). *Letters in Mathematical Physics*, **18**, 143.
Kulish, P. P., and Sklyanin, E. K. (1982). In *Lecture Notes in Physics*, Vol. 151, Springer, Berlin, p. 61.
Laughlin, R. B. (1983). *Physical Review Letters*, **50**, 1395.
Mansour, M. (1998). *International Journal of Theoretical Physics*, **37**, 2995.
Prange, P. E., and Girvin, S. M. (1990). *The Quantum Hall Effects*, Springer, New York.
Reshetkhin, N. Y., Takhtajan, L. A., and Fadeev, L. D. (1989).*Lening Mathematics Journal*, **1**, 177.
Sato, H. T. (1994). *Modern Physics Letters A*, **9**, 1819.
Sklyanin, E. K. (1991). *Functional Analysis and Its Applications*, **17**, 564.
Stone, M. (1992). *Quantum Hall Effect*, World Scientific, Singapore.
Weigmann, P. B., and Zabrodin, A. V. (1993a). cond-mat/9310017, Preprint.
Weigmann, P. B., and Zabrodin, A. V. (1993b). cond-mat/9312088, Preprint.